Stokes waves with constant vorticity: Numerical computation

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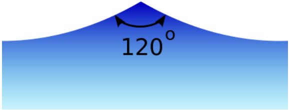
(Image from Teles da Silva and Peregrine (1988))

Stokes waves are

- traveling waves
- periodic
- at the surface of an incompressible inviscid fluid = water
- two dimensional
- acted on by gravity (no surface tension)
- infinitely deep or with a rigid flat bed

In the irrotational setting

Stokes conjectured



(Image from Wikipedia)

Amick, Fraenkel, and Toland (1982) proved that such a corner wave exists.

Recently, further advances — analytical and numerical — based on Babenko's equation:

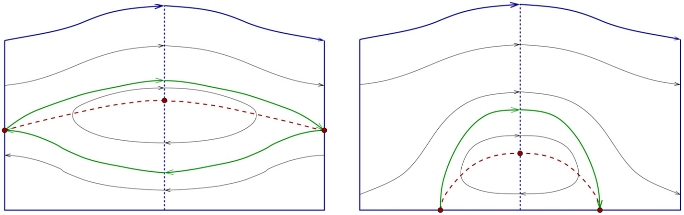
$$\boxed{\lambda^2 \mathscr{H} y' = y + y \mathscr{H} y' + \mathscr{H}(yy'),}$$

$y =$ fluid surface, $\lambda =$ Froude number, $\mathscr{H} =$ Hilbert transform.

In the rotational setting

Constantin and Strauss (2004) worked out global bifurcation for general vorticities.
Solutions do *not* permit critical layers, or internal stagnation.

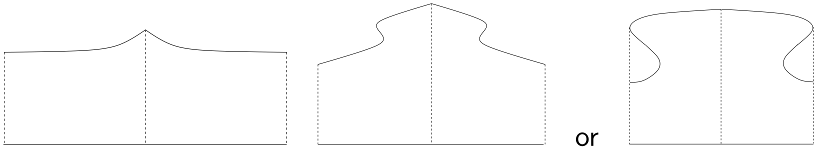
Wahlén (2009) observed them for **constant** vorticities.



For constant vorticities

Constantin, Strauss, and Varvaruca (2014) worked out global bifurcation, permitting overhanging, critical layers, and internal stagnation.

They conjectured that the limiting wave is:



By the way, the proof is *non*-constructive.

Our goal is to numerically study the conjecture.

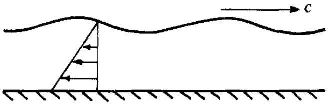
Earlier works include

- Simmen and Saffman (1985),
- Teles da Silva and Peregrine (1988),
- Vanden-Broeck (1994, 1996), ...

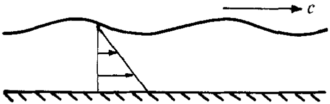
See also

- Vasan and Oliveras (2014),
- Ribeiro, Milewski, and Nachbin (2017),...

Formulation



downstream = − vorticity



upstream = + vorticity

(Figures from Teles da Silva and Peregrine (1983))

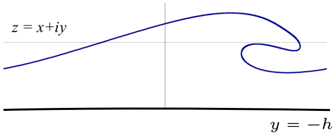
Let's write the velocity $(-\omega y - c, 0) + \nabla\phi$,
ω = constant vorticity, c = wave speed.

The problem is written:

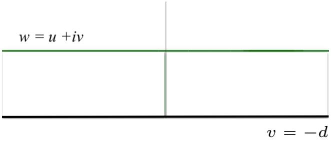
$$\begin{array}{|ll|} \hline \Delta\phi = 0 & \text{in fluid,} \\ \psi - \frac{1}{2}\omega y^2 - cy = 0 & \text{at surface,} \\ \frac{1}{2}(\phi_x - \omega y - c)^2 + \frac{1}{2}\phi_y^2 + gy = B & \text{at surface,} \\ \phi_y = 0 & \text{at bed,} \\ \hline \end{array}$$

ψ = harmonic conjugate of ϕ, B = Bernoulli constant.

Reformulation via conformal mapping



physical domain



conformal domain

The problem becomes

$$\boxed{\frac{(c+\omega(y+y\mathscr{T}y' - \mathscr{T}(yy')))^2}{(1+\mathscr{T}y')^2+(y')^2} = B + c^2 - 2gy.}$$

Definition. $\mathscr{T}(e^{iku}) = -i\coth(kd)e^{iku}$ for $k \neq 0$ an integer.
Formally, $\mathscr{T} \to \mathscr{H}$ as $d \to \infty$.

Reformulation to Babenko type

Use the fact:

$(\mathscr{T}+i)f$ is the boundary value of a holomorphic function in $-d<v<0$ whose imaginary part $=0$ at $v=-d$.

The problem is written:

$$\boxed{\begin{aligned}&(c^2+2B)\mathscr{T}y' - gy - c\omega y - g(y\mathscr{T}y' + \mathscr{T}(yy'))\\ &\quad -\tfrac{1}{2}\omega^2(y^2+\mathscr{T}(y^2y') + y^2\mathscr{T}y' - 2y\mathscr{T}(yy')) = \langle \text{LHS}\rangle\end{aligned}}$$

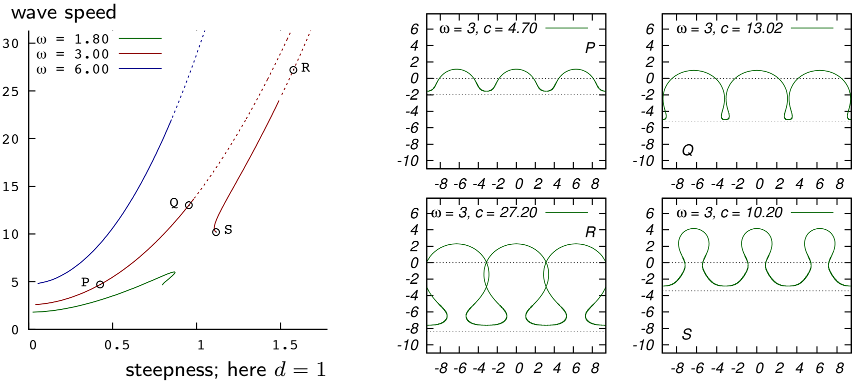
and

$$\boxed{\begin{aligned}&\langle(c+\omega(y+y\mathscr{T}y' - \mathscr{T}(yy')))^2\rangle\\ &\quad = \langle(B+c^2-2gy)((1+\mathscr{T}y')^2+(y')^2)\rangle\end{aligned}}$$

subject to

$$\boxed{\langle y(1+\mathscr{T}y')\rangle = 0.}$$

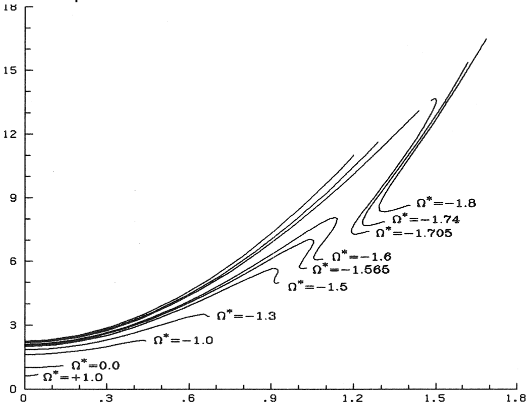
Sample waves



How to compare with earlier works?

We reproduce Simmen and Saffman (1985):

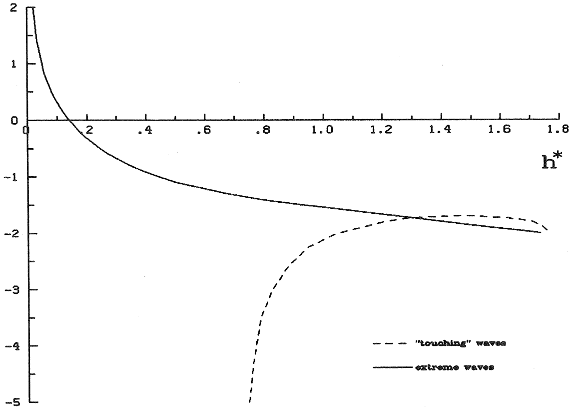
wave speed



steepness; here $d = \infty$

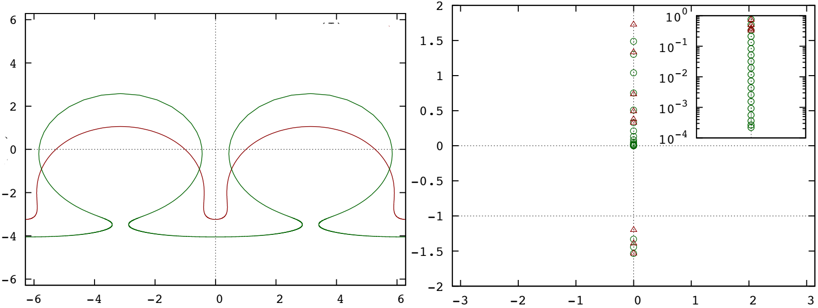
For Teles da Silva and Peregrine (1988), $d = \langle y \rangle + h$.

Limiting wave $\neq$ highest wave



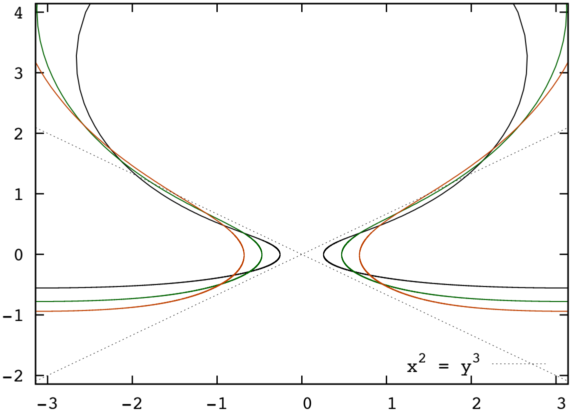
(Figure from Simmen's thesis (1984))

Effects of large positive vorticities



Waves (left) and singularities (right) for $\omega = 6$, $d = 1$ and $c = 15.80$ (red) and $c = 32.00$ (green).

New limiting wave?



Some open problems

- c is bounded throughout the solution curve?
- $c^2 + 2B \geqslant 0$ throughout the solution curve?
- Any C^1 solution is real analytic?